Preference Order in Group Decision-Making Through Fuzzy Partial Order Relations

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Abstract—This paper proposes a procedure for determining preference order using fuzzy partial order relation in circumstances of group decision-making. First, pairwise resemblance matrix is formulated, in which each element is derived from the difference between two opinions. Then, similar elements of the matrix are clustered logically, according to the properties of fuzzy similarity relation. After aggregating evaluations in each clustered subgroup, a priority matrix for these subgroups is constructed. Then, a preference matrix that has the nature of fuzzy partial order relation is derived from the priority matrix. In the construction process, the transitive law plays an important role in finding inconsistency of order. The fuzzy similarity relation enables one to investigate similarity and dissimilarity of opinions in the group at a time and the obtained fuzzy partial order relation can give preference order directly. The proposed procedure gives decision-makers lead a reasonable group decision-making in the context of logical treatment for various opinions due to diversified views or ideas.

Index Terms—Cluster of opinions, fuzzy partial order relation, fuzzy similarity relation, group decision-making, preference matrix

1. INTRODUCTION

Importance of decision-making problems often appear in our life, whether in public affairs or in personal events. When one faces some complicated matter, he may sort out problems, investigate cause-and-effect relationship, and seek possible actions. Thus he determines his appropriate action for the problem. In most cases, the optimal thing out of possible alternatives is chosen as the best action.

Numerous researchers have been coming to grips with modeling of human’s thought process and construction of logical way for obtaining final decision. Especially, in recent intricate society, the group decision-making is an important scheme for making a comprehensive definite policy and establishing a consensus of opinion. Decision-makers with their own lifestyles generally have a tendency to manage problems according to their unique sense of values and main interests. Therefore, in constructing group decision-support systems, our research interests lie in the problems how to categorize various opinions logically, based on similarity and dissimilarity and how to determine the ordering of preference effectively, making use of categorized opinions.

Analytic Hierarchy Process (AHP) [1] is well known as the procedure to solve multiple criteria decision-making problems. For the group decision-making, two aggregation ways have been presented [1], [2]. One is to aggregate each decision-maker’s pairwise comparison matrix at each phase appeared in the process. In the method, as individual opinions are aggregated on the way, the resultant group priority is given directly at the final stage. The other is to aggregate each member’s result at the final stage. That is, each member establishes his own evaluation individually, and then, final priority of the whole group is calculated by aggregating all evaluation results.

In order to calculate reciprocal weight among items, the geometric mean has been widely used in the former [3]-[5], while the arithmetic mean technique has been employed in the latter [6], [7] to aggregate individually evaluated result. In [8], taking the importance of similarity of individual perspective into account, the comprehensive aggregation procedure utilizing both arithmetic mean and geometric mean has been employed.

Decision-making group often consists of a lot of members who have various views and interests. Then, one sometimes adopts a technique to cluster homogeneous subgroup first, before deriving individual member’s evaluation value. In this connection, literatures [9], [10] have taken a stand on emphasizing the importance of homogeneity of evaluation when aggregating each opinion to get the final result.

Recent works [11], [12] have also assumed that the group members have various senses of values and ideas. Those methods divide group members into some homogeneous subgroups according to the similarity relation first, then, gathering priorities given by subgroups, final priority of the total group is aggregated. In [11], “cosine” derived from inner product of evaluation vectors is used as the way of expressing similarity. In fact, analyzing the difference in evaluation is significant in order to reach group consensus and must be taken into account in the decision-making process. Furthermore, “cosine” method has been improved in [12] by employing “sigmoid function”.

In this paper, a procedure for determining preference order in group decision-making is presented. Characteristic natures of fuzzy partial order relation play the important role in deriving preference order. First, a pairwise resemblance matrix whose elements are given by the difference between two opinions is introduced. The difference is expressed by the sigmoid function [12] that is useful to quantify similarity and dissimilarity. Then, the fuzzy similarity relation is utilized for categorizing pairwise resemblance matrix.
If the fuzzy similarity relation is not existed partly in the resemblance matrix and some unification is mandatory, a sort of negotiation is required among decision-makers. For the case, in this paper, the final value of nth power of the resemblance matrix is proposed as the consensus value. In fact, any fuzzy relation matrix has a nature that its transitive closure is transitive.

After aggregating evaluations in each clustered subgroup, a priority matrix for these subgroups is constructed. Then, a preference matrix that leads the preference order of the entire group directly is derived from the priority matrix. In deriving the matrix, the nature of fuzzy partial order relation is utilized. Especially, the transitive law plays an important role in finding inconsistency of order. The nature of fuzzy similarity relation is useful to investigate similarity and dissimilarity of opinions in the group at a time and the nature of fuzzy partial order relation can give preference order directly.

II. RESEMBLANCE OF OPINION VECTORS

Suppose that each evaluated opinion vector has been normalized as unit vector. Then, the difference of the two may be calculated by the angle between them. Conventional way of expressing a resemblance of two vectors is “cosine” derived from inner product. In this paper, considering that the cosine method has some drawback in symmetrizing similarity and dissimilarity, the resemblance is expressed by the sigmoid function rather than cosine function. Sigmoid function is shown to be useful to harmonize similarity and dissimilarity represented by fuzzy numbers[12].

A. Cosine from Inner Product

Let us consider two normalized vectors as shown in Fig.1. A resemblance expression of the two vectors \( \hat{F}_i \) and \( \hat{F}_j \) on the multi-dimensional space is derived from inner product [13].

\[
\cos \delta = \hat{F}_i \cdot \hat{F}_j.
\]

From (1), we can find relationship; If \( \hat{F}_i = \hat{F}_j \), then \( \cos \delta = 1 \), which implies “same”, and if \( \hat{F}_i \perp \hat{F}_j \), then \( \cos \delta = 0 \), which represents “different”.

In Fig. 2, case (a) denotes that the similarity of two vectors equals 1 which implies completely same, whereas dissimilarity is 0 that implies completely different. A problem appears for case (c). In case (c), despite the degree of similarity is given as 0.87, the degree of dissimilarity is calculated as 0.5. That is, although both (a) and (b) show well-balanced degree, i.e., 1+0=1 and 0+1=1, case (c) gives a value over 1, such that 0.87+0.5=1.37>1. When these numbers imply degree of fuzziness, this relationship is awkward, because “not a” is expressed by “1-a” in fuzzy theory.

B. Symmetric Evaluation Method

The symmetric evaluation method that improves number’s asymmetry utilizes sigmoid function. Fig.3 shows the sigmoid function that shows similarity and dissimilarity.

The combined sigmoid function \( S \) is denoted as [12]:

\[
S(f(\delta)) = \frac{1}{1 + e^{f(\delta)}},
\]

\[
f(\delta) = -\frac{1}{\tan 2\delta}.
\]

Where, the role of \( f(\delta) \) is a mapping from the angle-based expression to the number between minus infinity and plus infinity, such that \( f: [0, \pi/2] \rightarrow [-\infty, +\infty] \). These infinities are transformed to \([1,0]\) by the sigmoid function, using mapping \( S: [\infty, +\infty] \rightarrow [1,0] \). Using this function, we have \( S(\pi/6)=0.64 \) (similarity) and \( S(\pi/3)=0.36 \) (dissimilarity). In fact, this new similarity function gives well-balanced values of similarity against dissimilarity. Thus, the primary angle-based expression is well related to the fuzzy values \([1,0]\).

III. FUZZY SIMILARITY RELATION AND FUZZY PARTIAL ORDER RELATION

In this chapter, we describe fuzzy similarity relation and fuzzy partial order relation. Fuzzy similarity relation is the
extension of ordinary equivalence relation while fuzzy order relation is the extension of crisp partial order relation through fuzzy approach.

A. Fuzzy Similarity Relation

Let us consider the direct product of sets $X$ and $Y$, such that

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

Then fuzzy relation $R$ is characterized by a membership function $\mu_R$ as

$$\mu_R : X \times Y \rightarrow [0,1]$$

For (4), $R$ is composed as

$$R = [r_{ij}] = [\mu_R(x_i, y_j)], \ \forall x_i \in X, \forall y_j \in Y$$

Suppose $R$ and $T$ be relations in $X\times Y$ and $Y\times Z$, respectively. Then the composition of $R$ and $T$ denoted by $RT$ is defined as follows:

$$RT \Leftrightarrow \mu_{RT}(x,z) = \max_y \min \{\mu_R(x,y), \mu_T(y,z)\}$$

The next definition about fuzzy similarity relation gives the important role in this paper.

Definition 1

When the following three conditions are satisfied, the relation $R$ on set $X$ is called similarity relation[13],

(i) Reflexivity: $\mu_R(x,x) = 1$; $R \supseteq I$

(ii) Symmetry: $\mu_R(x,y) = \mu_R(y,x)$; $R = R^T$.

(iii) Transitivity: $\max_y \min \{\mu_R(x,y), \mu_R(y,z)\} \leq \mu_R(x,z)$;

or $RoR \subseteq R$

According to the nature of similarity relation, a group is logically categorized into some subgroups.

B. Mean Value and Aggregation

After categorization, it is necessary to aggregate opinion vectors in every subgroup. General way for the aggregation is to adopt mean value of items.

In this paper, as concerned opinion vectors are assumed to be normalized vectors, we adopt norm mean in aggregating evaluations.

The norm mean $\gamma_n$ for a vector $g = [g_1, g_2, \cdots, g_n]$ is defined as

$$\gamma_n = \left[ \frac{1}{n} \sum_{i=1}^{n} g_i^2 \right]^{1/2}$$

C. Fuzzy Partial Order Relation

Fuzzy partial order relation is mainly used in this paper, to make a preference order in decision-making.

Definition 3

For fuzzy relation $P$ to be a partial order relation on set $X$, the following conditions must be satisfied:

(i) Reflexivity: $\mu_P(x,x) = 1$; $P \supseteq I$

(ii) Anti-symmetry: $x \neq y$, $\mu_P(x,y) > 0 \rightarrow \mu_P(y,x) = 0$,

(iii) Transitivity: $\max_y \min \{\mu_P(x,y), \mu_P(y,z)\} \leq \mu_P(x,z)$;

In deriving preference order on alternatives, those natures of fuzzy partial order relation are utilized. Especially, (iii) is important to establish the right orders without contradiction. It is possible to visualize orders by Hasse diagram.

IV. CATEGORIZATION OF GROUP OPINIONS

We discuss the possibility of categorizing diversified opinions in this chapter. Usually, when two evaluated vectors are presented, the judgment on similarity and dissimilarity is left in decision maker’s subjectivity or experience.

A. Composition of Fuzzy Resemblance Matrix

According to the discussion in II, we employ the sigmoid function $S(\delta_i) : [0, \pi/2] \rightarrow [1,0]$, as the similarity function. Here, $S(\delta_i) = 1$ implies completely similar and $S(\delta_i) = 0$ indicates dissimilar. Thus a fuzzy resemblance matrix can be defined as follows:

Definition 4

For the normalized opinion vectors $F_i$ and $F_j$, fuzzy relation matrix in which elements are derived from angles between two opinion vectors as $S(\delta_{ij})$;

$$S = [S_{ij}], \ S_{ij} = S(\delta_{ij}), \ i,j = 1,2,\ldots,n$$

is called resemblance matrix.

Now let consider the uniting rule for three opinion vectors. Suppose evaluated three vectors $F_i$, $F_j$, and $F_k$ given by three persons $I$, $J$, and $K$. Again we assume $|F_i|$, $|F_j|$ and $|F_k|$ equal 1. There exists some rule among $S_{ik}$, $S_{ij}$ and $S_{kj}$ for the resemblance matrix $S$ in Definition 4 to be a fuzzy similarity relation. That is, $S_{ik}$, $S_{ij}$ and $S_{kj}$ must have the relationship so that $S$ satisfies three conditions given in Definition 1. This gives the uniting rule for three opinion vectors.

Thus we have the following important theorem [12] referring to Definitions 1 and 4.

Theorem 1

For three normalized opinion vectors, resemblance matrix $S$ is the similarity relation under the condition
\[ S_{ik} \geq S_{ji} = S_{kj}, \quad \text{for } j=1,2,3 \quad (9) \]

Proof

Conditions for reflexivity and symmetry are clearly satisfied from definition of \( S_{ij} = S(\delta_{ij}) \). Therefore we prove the transitivity law:

\[ S_{ik} \geq \vee (S_{ij} \wedge S_{kj}), \quad \text{for } j=1,2,3 \quad (10) \]

In (10), \( \vee \) and \( \wedge \) denote “max” and “mini” operations, respectively.

A general term \( S_{ij} \wedge S_{kj} \) in the right-hand side can be rewritten as \( S_{ji} \wedge S_{kj} \), because \( S \) is symmetric. Hence, using (9), we have

\[ S_{ik} \geq S_{ji} \wedge S_{kj} \quad \text{for } j=1,2,3 \quad (11) \]

Thus (10) is clearly satisfied. \( \Box \)

From the definition of \( S_{ij} \), \( S \) in (8) generally satisfies reflexivity and symmetry properties. Therefore, theorem 1 concerning the transitivity is important to pass judgment on parameters whether associable or not.

B. Categorization and Similarity

When the resemblance matrix has the property of fuzzy similarity relation, all opinions can be included in a group. However, in most cases, only some parts satisfy the similarity. Then, the group is clustered into some subgroups with similar opinions by recognizing satisfying parts of the matrix. Otherwise, some of decision-makers may change their opinions according to the negotiation etc. That kind of choice depends on decision-makers who are in the responsibility of establishing the plan.

C. Threshold Value

When the resemblance matrix satisfies the property of fuzzy similarity relation, a logical categorization is certainly possible. In actual decision-making circumstances, however, the idea of threshold is sometimes significant.

For example, suppose a resemblance matrix given by

\[
S = \begin{bmatrix}
1 & 0.9 & 0.3 \\
0.9 & 1 & 0.3 \\
0.3 & 0.3 & 1
\end{bmatrix}
\]

For the case, possible threshold values \( a, b \) and \( c \) may be given as \( a \geq 0.9 \), \( 0.3 \leq b < 0.9 \) and \( c < 0.3 \), respectively. If decision-makers choose \( c \) as the threshold, all opinions can be united together and recognized as similar. Logical consistency is certainly existed for the case. However, it may be unreasonable to regard 0.3, 0.9 and 1 as similar, from our usual sense. One may choose rather \( a \) than \( c \). A method of giving the threshold value, taking human’s sense, has been discussed in [12].

D. Adjustment of Opinions

When the transitivity of classification matrix is not satisfied, it is impossible to discuss the logical similarity. Hence, if resemblance matrix \( S \) does not satisfy similarity relation, one may utilize next property [14] for obtaining consensus value:

Property 1

For any fuzzy relation \( S \), the transitive closure of \( S \), denoted by \( S^* \) in (12), is transitive:

\[
S^* = S \cup S^2 \cup \cdots \cup S^n
\]

If some opinion is breaking logical consistency of group opinion, the result of convergence matrix \( S^* \) may give a useful suggestion in the negotiating environment.

IV. CONSTRUCTION OF PREFERENCE ORDER MATRIX

After aggregating evaluations in each subgroup, we need to construct a priority matrix for these subgroups. Elements of the matrix are calculated using the norm mean presented in (7). Then, a preference matrix that leads the preference order of the entire group will be derived from the priority matrix. In deriving the matrix, nature of fuzzy partial order relation is utilized.

A. Priority Matrix

Suppose that \( m \) members belong to subgroup \( a \), then, a evaluation vector \( Q_a \) given by these members for alternative \( l \) is represented as

\[
Q_a^l = \{ q_{i1}^l, q_{i2}^l, \cdots, q_{im}^l \}
\]

Then, an aggregation of each value is carried out by the norm mean in (7). Hence, the entire evaluation of subgroup \( a \) for \( l \) is calculated by

\[
\lambda_a^l = | Q_a^l | = \{ (1/m) \sum_{k=1}^m (q_{ik}^l)^2 \}^{1/2}
\]

If the numbers of subgroups and alternatives are \( \alpha \) and \( \beta \), respectively, we can obtain group-alternative matrix \( \Delta \) as

\[
\Delta = [ \lambda_{ij} ], \quad i=1,2,\ldots,\alpha, \quad j=1,2,\ldots,\beta
\]

The value of \( \lambda_{ij} \) shows that subgroup \( i \) estimates alternative \( j \) as \( \lambda_{ij} \). Law vector gives all alternative’s score. On the other hand, Law vector of \( \Delta^T \) denotes all scores estimated by subgroups.
Now let the \(i\)th row vector of \(\mathbf{A}\) be \(\mathbf{\lambda}_i\), that includes all alternative’s score estimated by group \(i\). Then, we derive the priority matrix \(\mathbf{\Delta}_i = [\delta_{ij}]\) for \(\mathbf{\lambda}_i = [\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{i\beta}]\), as

\[
\mathbf{\Delta}_i = \begin{pmatrix}
\lambda_{i1}/\lambda_{i2}, & \lambda_{i1}/\lambda_{i3}, & \ldots, & \lambda_{i1}/\lambda_{i\beta} \\
\lambda_{i2}/\lambda_{i1}, & \lambda_{i2}/\lambda_{i3}, & \ldots, & \lambda_{i2}/\lambda_{i\beta} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{i\beta}/\lambda_{i1}, & \lambda_{i\beta}/\lambda_{i2}, & \ldots, & \lambda_{i\beta}/\lambda_{i\beta}
\end{pmatrix}
\]

Diagonal elements in \(\mathbf{\Delta}_i\) are found to be 1 and each element gives ratio between estimated values. Also, elements have a reciprocal nature that if \(\lambda_{ij}/\lambda_{ik} > 1\), then \(\lambda_{ij}/\lambda_{ik} < 1\).

### B. Preference Order Matrix

A preference order matrix

\[ \Phi_i = [\phi_{ij}], \; k=1,2,\ldots,\beta, \; j=1,2,\ldots,\beta \]  

is derived from the aforementioned priority matrix \(\mathbf{\Delta}_i\), by making replacements as

\[ (p1) \phi_{ij} = \delta_{ij}, \; \text{for} \; k, j=1,2,\ldots,\beta, \]

\[ (p2) \phi_{ij} = 0, \; \text{if} \; \delta_{ij} = \lambda_{ij}/\lambda_{ik} > 1 \; \text{or} \; \delta_{ij} = \delta_{jk} \]

Note that the replacement does not affect the original ordering of \(\Phi_i\) and \(\mathbf{\Delta}_i\). In this way, we can derive a preference matrix that satisfies properties:

(i) Reflexivity : \(\mu_p(x,x) = 1\),

(ii) Anti-symmetry : \(x \neq y; \mu_p(x,y) > 0 \rightarrow \mu_p(y,x) = 0\).

If the matrix \(\Phi_i\) has satisfied property of “transitivity”, it becomes the fuzzy partial order relations.

### C. Combination of Preference Order Matrix

In section B, we have derived a preference order matrix for one subgroup. In this section, some preference order matrices are combined to reach final evaluation.

In order to make discussions simple, we suppose that two subgroups \(A\) and \(B\) give evaluations for alternatives I, II and III. Then we assume two Hasse diagrams are obtained from fuzzy partial order matrices as shown in Fig.4.

Again, remember that numbers are preference ratio between two items.

The total evaluation is developed from these two diagrams. For the sake of convenience, in this paper, Fig. 4 is redrawn using oriented graphs shown in Fig.5.

Fig.5 Oriented graphs

We assume that the maximum evaluation value allowed for every subgroup is \(\lambda_m\), then, scores estimated for I, II, III in group A are given by \(\lambda_{m1}, 0.2\lambda_{m1}, \text{and } 0.4\lambda_{m1}\) respectively. On the other hand, scores of I, II, III in B are recognized as \(0.6\lambda_{m1}, \lambda_{m1}, 0.18\lambda_{m1}\). Therefore, in this example, final scores estimated for alternatives I, II, and III become

\[
\begin{pmatrix}
V_1 \\
V_2 \\
V_3
\end{pmatrix} = \begin{pmatrix}
\lambda_{m1} + 0.6\lambda_{m1} \\
0.2\lambda_{m1} + \lambda_{m1} \\
0.4\lambda_{m1} + 0.18\lambda_{m1}
\end{pmatrix} = \begin{pmatrix}
1.6\lambda_{m1} \\
1.2\lambda_{m1} \\
0.58\lambda_{m1}
\end{pmatrix}
\]

Sometimes, special subgroup may be weighted by some reason, i.e. social or political reason etc. For the case, score of alternative \(i\) is calculated from the next equation, considering weights of subgroups:

\[
V_i = (\phi^{A}_{ki} \cdot w^A + \phi^{B}_{ki} \cdot w^B + \phi^{C}_{ki} \cdot w^C) \lambda_m \tag{18}
\]

In (18), \(\phi^{T}_{ki}, (T=A,B,C)\) is an element appeared in preference order matrix in group \(T\). Also, \(w^A, w^B, \text{and } w^C\) have been defined as weights of group \(A, B\) and \(C\), respectively, and \(k\) is a top node connected to node \(i\).

### V. FLOW OF PROCEDURE FOR DECISION-MAKING

Fig.6 shows the proposed algorithm for leading preference order in decision-making circumstance. Procedure at each step is as follows:

Step 1: Obtain crude opinion vectors \(\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots, \mathbf{\Phi}_n\) from all decision-makers who join to solve the problem. Opinion vectors may be calculated by AHP etc..

Step 2: Develop resemblance matrix \(S\) in which elements are derived from angles between two opinion vectors as \(s(\delta_{ij})\).

Step 3: Check if \(S\) has the nature of similarity relation. If yes, go to Step 4. Otherwise, go to Step 5.

Step 4: Do logical clustering processing to categorize a group into some subgroups.

Step 5: When no grouping is possible, they need modification of opinions.
In this paper, a method for determining preference order in group decision-making circumstances was presented. Characteristic nature of fuzzy partial order relation plays the important role in deriving preference order of groups. First, a pairwise resemblance matrix whose elements are obtained by the difference between two opinions is constructed. The difference is expressed, in this paper, by the sigmoid function technique [12] that is useful to quantify similarity and dissimilarity. Then, the fuzzy similarity relation is utilized for categorizing pairwise resemblance matrix.

When the fuzzy similarity relation is not existed partly in the resemblance matrix and the some unification is mandatory, a sort of negotiation is required among decision-makers. For the case, in this paper, the final value of nth power of the resemblance matrix is proposed as the consensus value. In fact, any fuzzy relation matrix has an important nature that its transitive closure is transitive.

After aggregating evaluations in each clustered subgroup, a priority matrix for these subgroups is constructed. Then, a preference order matrix that leads the preference order of the entire group directly is derived from the priority matrix. In deriving the matrix, the nature of fuzzy partial order relation is utilized. Especially, the transitive law plays the important role in finding inconsistency of order. The nature of fuzzy similarity relation is useful to investigate similarity and dissimilarity of opinions in the group at a time and the nature of fuzzy partial order relation can give preference order directly.

The proposed method makes us possible to examine and regulate diverse opinions and ideas in the context of group decision-making scenario.

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