Abstract—White noise deconvolution or input white noise estimation problem has important application background in oil seismic exploration. Based on the optimal information fusion rules weighted by matrices, diagonal matrices and scalars in the linear minimum variance sense, three distributed fused white noise deconvolution smoothers are presented for the linear discrete time-varying stochastic control systems with multisensor and colored measurement noises. The accuracy of the fuser with the matrix weights is higher than that of the fuser with scalar weights, but the computational burden of the fuser with the matrix weights is larger than that of the fuser with scalar weights. The accuracy and computational burden of the fuser with diagonal matrix weights are between both of them. They are locally optimal, and globally suboptimal. Their accuracy is higher than that of local white noise estimators. They can handle the white noise fused filtering and smoothing problems. In order to compute the optimal weights, the new formula of computing the local estimation error covariances is given. A Monte Carlo simulation example for a Bernoulli-Gaussian input white noise fused smoother shows their effectiveness.

Index Terms—Time-varying system, colored measurement noises, multisensor information fusion, white noise estimators, deconvolution, reflection seismology

1. INTRODUCTION

In oil seismic exploration [1–4], an explosive is detonated below the earth’s surface, so that the seismic waves generated, and are reflected in different geological layers. The oil exploration is performed via the reflection coefficient sequence, which can be described by Bernoulli-Gaussian white noise, which is the input signal of receivers. Estimating input white noise is called deconvolution, which has been of important application value for finding and discovering the oil field and determining its geometry shape. White noise estimators also occur in many fields including communication, signal processing, and state estimation. Mendel [1–3] and Kornmylo [4] presents the optimal input white noise estimators with application to oil seismic exploration based on the Kalman filter, but the measurement white noise estimators was not presented. Deng, Zhang, Liu, and Zhou [5] presents a unified white noise estimation theory based on the modern time series analysis method, which not only includes input noise estimators but also includes measurement white noise estimators. But its limitation is that it can only solve the steady-state white noise estimators but cannot solve the optimal white noise estimation problem for time-varying system. The general and unified white noise estimation theory based on Kalman filtering for time-varying systems has been presented [6,7].

Recently, Sun [8] gives the optimal information fusion white noise filter weighted by scalars based on Kalman predictor, but it doesn’t solve the information fusion white noise smoothing problems. Sun's white noise fused filter is not suitable for applications. For example, for multisensor system with uncorrelated noises, Sun's white noise fused filter becomes zero, whose accuracy is lower. In order to improve the fuser accuracy, we must consider the white noise fused smoother. Although Sun [9] presents a white noise fused smoother, but it is suitable only for time-varying systems with white measurement noise, and only gives a white noise fused smoother with matrix weights, and doesn’t present the steady-state white noise fused smoother.

The multisensor information fusion has received great attention in recent years due to extensive application backgrounds, which has widely been applied to many field including guidance, defence, robotics, integrated navigation, target tracking, GPS positioning, and signal processing. For Kalman filtering-based fusion, two basic fusion methods are centralized and decentralized (or distributed, or weighted) fusion methods, depending on whether raw data are used directly for fusion or not [10]. The centralized fusion method can give the globally optimal state estimation by directly combing the local measurement data, but its disadvantages are that it may require a larger computational burden and high data rates for communication. The distributed fusion method can give the globally optimal or suboptimal state estimation by weighting the local state estimators. This method has considerable advantages: it can facilitate fault detection and isolation more conveniently, and can increase the input data rates significantly.

The weighted fusion approach is an important distributed fusion approach. It is quite important how to select the optimal weighting rules and how to compute the cross-covariances among local estimation errors. The weighted least squares
(WLS) fusion rule was presented by Carlson [11]. The maximum likelihood (ML) fusion rule was given by Kim [12]. The optimal fusion rules weighted by matrices, diagonal matrices, and scalars have been presented in linear minimum variance sense [8,13,14]. Notice that the optimal fusion estimate is related to the performance index of optimization, and is within a restricted local linear space. All these fusion rules give locally optimal estimators, which are globally suboptimal compared with the centralized fusion estimator. So far, the information fusion is mainly focused on the state estimation problems, but the information fusion concerning the input white noise is seldom reported, which has an important application background in oil seismic exploration [1-4].

In order to overcome the above drawback and limitation, the unified and general optimal information fusion white noise deconvolution estimators weighted by matrices, diagonal matrices and scalars are presented for time-varying systems with multisensor and colored measurement noises in this paper, which includes Sun [8]’s results as a special case. They can handle the white noise fused filtering, smoothing and prediction problems. In order to compute the optimal weights, the formula computing the local estimation error covariances is presented, which is completely different from Sun’s formula in [9]. As a special case, the distributed fusion steady-state white noise estimators are also presented.

II. PROBLEM FORMULATION

Consider the discrete time-varying linear stochastic control system with L sensors

\[ x(t+1) = \Phi(t)x(t) + B(t)u(t) + \Gamma(t)w(t) \]
\[ z_i(t) = \overline{H}_i(t)x(t) + \eta_i(t), i = 1, \ldots, L, \]
\[ \eta_i(t+1) = A_i(t)\eta_i(t) + \xi_i(t) \]

where \( t \) is discrete time, \( x(t) \in \mathbb{R}^n \) is the state, \( z_i(t) \in \mathbb{R}^m, i = 1, \ldots, L, \) are the measurements, \( u(t) \in \mathbb{R}^p \) is the known control input, \( w(t) \in \mathbb{R}^r \) and \( \xi_i(t) \in \mathbb{R}^\omega \) are the white noises, \( \eta_i(t) \in \mathbb{R}^m \) are the colored measurement noises, and \( \Phi(t), \Gamma(t), \overline{H}_i(t) \) and \( A_i(t) \) are time-varying matrices with compatible dimensions.

**Assumption 1.** \( w(t) \) and \( \xi_i(t) \) are independent white noises with zero mean and variance matrix \( Q(t) \) and \( Q_i(t) \), respectively.

**Assumption 2.** The initial state \( x(0) \) with mean \( \mu \) and error variance matrix \( P_0 \), is uncorrelated with \( w(t) \) and \( \xi_i(t) \), \( i = 1, \ldots, L. \)

The optimal information fusion white noise deconvolution smoother problem is to find the optimal (linear minimum variance) fusion white noise deconvolution smoothers \( \hat{w}_0(t|1+N), N > 0 \) weighted by matrices, diagonal matrices and scalars based on the local white noise deconvolution smoothers \( \hat{w}_i(t|1+N) \), respectively.

Introducing new measurements

\[ y_i(t) = z_i(t+1) - A_i(t)\eta_i(t) - \overline{H}_i(t+1)B(t)u(t) \]

Substituting (1)-(3) into (4), we have

\[ y_i(t) = (\overline{H}_i(t+1)\Phi(t) - A_i(t)\overline{H}_i(t))x(t) + \overline{H}_i(t+1)\Gamma(t)w(t) + \xi_i(t) \]

Setting

\[ H_i(t) = \overline{H}_i(t+1)\Phi(t) - A_i(t)\overline{H}_i(t) \]
\[ v_i(t) = \overline{H}_i(t+1)\Gamma(t)w(t) + \xi_i(t) \]

and combining (1) and (5), we have

\[ x(t+1) = \Phi(t)x(t) + B(t)u(t) + \Gamma(t)w(t) \]
\[ y_i(t) = H_i(t)x(t) + v_i(t), i = 1, \ldots, L \]

The new system (8) and (9) are the linear stochastic control system with correlated white noises \( w(t) \) and \( v_i(t), i = 1, \ldots, L \), and with correlated measurement noises \( v_i(t), i = 1, \ldots, L \), that is

\[ E\left[ \begin{array}{c} w(t) \\ v_i(t) \end{array} \right] = \begin{bmatrix} Q(t) \\ S_i(t) \end{bmatrix}, E\left[ \begin{array}{c} w(t) \end{barray} \right] = \begin{bmatrix} R(t) \\ R_i(t) \end{bmatrix} \]

\[ R(t) = \overline{H}_i(t)\Gamma(t)Q(t)\Gamma(t)\overline{H}_i^T(t) + Q_i(t) \]

\[ S_i(t) = Q(t)\Gamma(t)\overline{H}_i^T(t) \]

\[ R_i(t) = \overline{H}_i(t)\Gamma(t)Q(t)\Gamma(t)\overline{H}_i^T(t) \]

where \( R_i(t) = R_i(t), E \) is the expectation, the superscript T denotes the transpose, and \( \delta_{ik} \) is the Kronecker delta function, \( \delta_{ik} = 1, \delta_{ik} = 0 \).

III. DISTRIBUTED FUSION WHITE NOISE DECONVOLUTION ESTIMATORS

**Lemma 1**[14]. For the multisensor time-varying systems (1)-(3) with the assumptions 1 and 2, the ith sensor subsystem has the local Kalman predictor for the state as

\[ \dot{x}_i(t+1|1) = \Phi(t)\dot{x}_i(t|1) + B(t)u(t) + K_{pi}(t)e_i(t) \]

\[ e_i(t) = y_i(t) - H_i(t)\dot{x}_i(t|1) \]

or

\[ \dot{x}_i(t+1|1) = \Phi(t)\dot{x}_i(t|1) + B(t)u(t) + K_{pi}(t)y_i(t) \]

\[ \dot{e}_i(t) = \Phi(t)\dot{e}_i(t) + \Gamma(t)S_i(t)Q_i^{-1}(t) \]

\[ Q_i(t) = H_i(t)P_i(t|1)H_i^T(t) + R_i(t) \]

where prediction error variance matrix \( P_i(t|1) \) satisfy the Riccati equation

\[ P_i(t|1) = \Phi(t)P_i(t|1)\Gamma(t) - \Phi(t)P_i(t|1)H_i^T(t) + \Gamma(t)S_i(t)H_i(t)P_i(t|1)H_i^T(t) + R_i(t) \]

648
\[ H^T(t) + \Gamma(t)S_j(t) = \Gamma(t)R_j(t) \]  
with initial value \( \hat{x}_i(0) = 0 \), \( P_i(0) = 0 \).

**Proof.** The proof of Lemma 1 is given in [14], which is omitted.

**Theorem 1.** For the multisensor time-varying system (1)-(3) with the assumptions 1 and 2, the cross-covariance matrices among local prediction errors are given as

\[
P_{ij}(t+1|t) = \Psi_{ji}(t)P_{ij}(t|t-1)\Psi^T_{ji}(t) + \Gamma(t)S_j(t) - K_{pi}(t)Q_{ji}(t) + K_{pi}(t)R_j(t)K_{pi}^T(t)
\]

where \( i, j = 1, \ldots, L \), or

\[
P_{ij}(t+1|t) = \Psi_{ji}(t)P_{ij}(t|t-1)\Psi^T_{ji}(t) + \Gamma(t)S_j(t) - K_{pi}(t)Q_{ji}(t) + K_{pi}(t)R_j(t)K_{pi}^T(t)
\]

specially, when \( i = j \) (18) becomes (16), where we define

\[
P_{ii}(t+1|t) = P_i(t+1|t) \text{, with the initial value } P_i(0) = 0, i = 1, \ldots, L
\]

**Proof.** From [14], we have the prediction error equation

\[
\hat{\hat{x}}_i(t+1) = \Psi_{ji}(t)\hat{\hat{x}}_j(t|t-1) + \Gamma(t)w(t) - K_{pi}(t)v_i(t)
\]

where \( w(t) \) is uncorrelated with \( v_i(t) \) and \( \hat{\hat{x}}_i(t|t-1) \). Using (20) yields (17).

**Lemma 2 [14].** For the multisensor time-varying system (1)-(3) with the assumptions 1 and 2, the \( i \)th sensor subsystem has the local optimal white noise deconvolution estimators

\[
\hat{\hat{w}}_i(t|t+N) = 0, N < 0, i = 1, \ldots, L
\]

where we define

\[
M_j(t|t) = S_j(t)Q_{ji}^{-1}(t),
\]

\[
M_j(t|t+1) = D_j(t)H_j^T(t+1)Q_{ji}^{-1}(t+1),
\]

\[
M_j(t|t+j) = D_j(t)\left(\prod_{k=0}^{j-1} \Psi_{ji}^T(t+k)\right)H_j^T(t+j)Q_{ji}^{-1}(t+j), j > 1
\]

where we define:

\[
D_j(t) = Q(t)\Gamma(t) - S_j(t)K_{pi}^T(t)
\]

and its variance

\[
P^w_j(t|t+N) = \mathbb{E}(\hat{\hat{w}}_i(t|t+N))\hat{\hat{w}}_i^T(t|t+N) \text{ is given as}
\]

\[
P^w_j(t|t+N) = Q(t)\sum_{j=0}^{N} M_j(t|t+j)Q_{ji}(t|t+j)M_j^T(t|t+j), N \geq 0,
\]

**Proof.** The proof of Lemma 2 is given in [14], which is omitted.

**Theorem 2.** For the multisensor time-varying system with the assumptions 1 and 2, the cross-covariance matrices among local estimation error cross-variances

\[
P^w_{ij}(t|t+N) = \Psi_{ji}(t)P_{ij}(t|t-1)\Psi^T_{ji}(t) + \sum_{p=0}^{N} [K_{pi}(t), K_{pj}(t)] \begin{bmatrix} Q(t+p) & S_j(t+p) & K_{pj}^T(t) \\ S_j^T(t+p) & R_j(t+p) & K_{pj}^T(t) \end{bmatrix}
\]

where \( i, j = 1, \ldots, L \), and we define

\[
P^w_{ii}(t|t+N) = \sum_{k=0}^{N} M_i(t|t+k)H_i(t+k)\Psi_{ji}(t|t+k,t)
\]

\[
K_{pi}(t) = \delta_{pj}I_p - \sum_{k=p+1}^{N} M_j(t|t+k)H_j(t+k)\Psi_{ji}(t|t+k,t)
\]

\[
\Psi_{ji}(t) = \sum_{k=0}^{N} M_i(t|t+k)H_i(t+k)\Psi_{ji}(t|t+k,t)
\]

where \( w(t) \) is uncorrelated with \( v_i(t) \) and \( \hat{\hat{x}}_i(t|t-1) \). Using (20) yields (17).

**Proof.** From [14], we have the prediction error equation

\[
\hat{\hat{x}}_i(t+1) = \Psi_{ji}(t)\hat{\hat{x}}_j(t|t-1) + \Gamma(t)w(t) - K_{pi}(t)v_i(t)
\]

where we define

\[
M_j(t|t) = S_j(t)Q_{ji}^{-1}(t),
\]

\[
M_j(t|t+1) = D_j(t)H_j^T(t+1)Q_{ji}^{-1}(t+1),
\]

\[
M_j(t|t+j) = D_j(t)\left(\prod_{k=0}^{j-1} \Psi_{ji}^T(t+k)\right)H_j^T(t+j)Q_{ji}^{-1}(t+j), j > 1
\]

where we define:

\[
D_j(t) = Q(t)\Gamma(t) - S_j(t)K_{pi}^T(t)
\]

the local error \( \hat{\hat{w}}_i(t|t+N) = w(t) - \hat{\hat{w}}_i(t|t+N) \) and its variance

\[
P^w_j(t|t+N) = \mathbb{E}(\hat{\hat{w}}_i(t|t+N)\hat{\hat{w}}_i^T(t|t+N)) \text{ is given as}
\]

\[
P^w_j(t|t+N) = Q(t)\sum_{j=0}^{N} M_j(t|t+j)Q_{ji}(t|t+j)M_j^T(t|t+j), N \geq 0,
\]

**Proof.** The proof of Lemma 2 is given in [14], which is omitted.
\[ \tilde{u}_i(t | t + N) = w(t) - \sum_{k=0}^{N} M_i(t | t + k) H_i(t + k) \times \Psi_p(t + k, t) \tilde{x}_i(t | t - 1) - \sum_{k=1}^{N} \sum_{\gamma=1}^{k} M_i(t | t + k) H_i(t + k) \Psi_p(t + k, t + \gamma) \times \Gamma(t + \gamma - 1) w(t + \gamma - 1) + \sum_{k=1}^{N} \sum_{\gamma=1}^{k} M_i(t | t + k) H_i(t + k) \Psi_p(t + k, t + \gamma) \times K_p(t + \gamma - 1) v_i(t + \gamma - 1) - \sum_{k=1}^{N} M_i(t | t + k) v_i(t + k) \] (35)

Combining the terms in (35) for \( \tilde{x}_i(t | t - 1) \), \( w(t + p) \), and \( v_i(t + p) \), respectively, we have

\[ \tilde{w}_i(t | t + N) = \Psi_{\theta_i}(t) \tilde{x}_i(t | t - 1) + \sum_{p=0}^{N} K_{ip}^w(t) w(t + p) + \sum_{p=0}^{N} K_{ip}^v(t) v_i(t + p) \] (36)

or

\[ \tilde{x}_i(t | t + N) = \Psi_{\theta_i}(t) \tilde{x}_i(t | t - 1) + \sum_{p=0}^{N} \left[ K_{ip}^w(t) K_{ip}^v(t) \right] \begin{bmatrix} w(t + p) \\ v_i(t + p) \end{bmatrix} \] (37)

Noting that \( \tilde{x}_i(t | t - 1) \) is uncorrelated with \( w(t + p) \) and \( v_i(t + p), p = 0, \cdots, L \). Using (10) and (37), we obtain (26) - (30).

**Theorem 3.** For the multisensor time-varying system (1)-(3) with the assumptions 1 and 2, three distributed optimal information fusion white noise estimators as

\[ \tilde{w}_0(t | t + N) = \sum_{i=1}^{L} \Omega_i(t) \tilde{w}_i(t | t + N), N \geq 0 \] (38)

\[ \tilde{w}_0(t | t + N) = 0, N < 0 \] (39)

For the fuser with matrix weights, we have

\[ [\Omega_1(t), \cdots, \Omega_L(t)] = (e^T P^{-1}(t | t + N) e)^{-1} e^T P^{-1}(t | t + N) \]

(40)

\[ P(t | t + N) = (P_0(t | t + N))_{\text{local}, i}, j = 1, \cdots, L \] (41)

where \( e^T = [I_a, \cdots, I_a] \), and the fused error variance matrix \( P_0^{w*}(t | t + N) \) is given as

\[ P_0^{w*}(t | t + N) = [e^T P^{-1}(t | t + N) e]^{-1} \] (42)

For the fuser with scalar weights, \( \Omega_i(t) = \omega_i(t) \), we have

\[ [\omega_1(t), \cdots, \omega_L(t)] = (e^T P(t | t + N) e)^{-1} e^T P^{-1}(t | t + N) \] (43)

\[ P(t | t + N) = (tr P_0^{w*}(t | t + N))_{\text{local}, i}, j = 1, \cdots, L \] (44)

where \( tr \) denotes the trace of matrix, \( e^T = [1, \cdots, 1] \) and the fusion error variance matrix \( P_0^w(t | t + N) \) is given as

\[ P_0^w(t | t + N) = \sum_{i=1}^{L} \omega_i(t) \omega_i(t) P_0^{w}(t | t + N) \] (45)

For the fuser with diagonal matrix weights, we have

\[ \Omega_i(t) = (\text{diag} [\omega_{i1}(t), \cdots, \omega_{iL}(t)]) \]

(46)

\[ [\omega_{i1}(t), \cdots, \omega_{iL}(t)] = (e^T P^{-1}(t | t + N) e)^{-1} e^T P^{-1}(t | t + N) \] (47)

\[ P_0(t | t + N) = (P_0^{w}(t | t + N))_{i | i}, j = 1, \cdots, L \] (48)

\[ P_0(t | t + N) = (P_0^{w}(t | t + N))_{i | i}, j = 1, \cdots, L \] (49)

where \( e^T = [1, \cdots, 1] \), and \( P_0^{w}(t | t + N) \) are the \((i,i)\) th diagonal elements of \( P_0^{w}(t | t + N), k, j = 1, \cdots, L, i = 1, \cdots, n \).

The trace of the fused error variance matrix \( P_0^{w*}(t | t + N) \) is given as

\[ tr P_0^{w*}(t | t + N) = \sum_{i=1}^{n} [e^T P^{-1}(t | t + N) e]^{-1} \] (50)

Denoting the centralized fusion error variance matrix as \( P_0^{wc}(t | t + N) \), we have the accuracy the relation

\[ tr P_0^{wc}(t | t + N) \leq tr P_0^{w*}(t | t + N) \leq tr P_0^{w}(t | t + N) \leq tr P_0^{w}(t | t + N) \leq tr P_0^{w}(t | t + N) \] (51)

**Proof.** Applying the three optimal fusion formulas weighted by matrices, diagonal matrices and scalars in [9], we directly obtain Theorem 3.

The equation (50) shows that the obtained three weighted fusers are locally optimal, but are globally suboptimal, and their accuracy is lower than that of the centralized fuser, and is higher than that of each local estimator. The accuracy of the fuser with matrix weight is higher than that of the fuser with scalar weight, and the accuracy of the fuser with diagonal matrix weight is between both of them.

**IV. STEADY-STATE FUSION WHITE NOISE DECONVOLUTION ESTIMATOR**

For the time-variant system (1)-(3) with constant matrices

\[ \Phi(t) = \Phi, B(t) = B, \Gamma(t) = \Gamma, \bar{H}_i(t) = \bar{H}_i, A_i(t) = A_i, Q(t) = Q, \bar{Q}_e(t) = \bar{Q}_e, \]

we have \( H(t) = H, S_i(t) = S_i, R_i(t) = R_i, \) and \( R_{ij}(t) = R_{ij} \). If every local sensor subsystem has steady-state Kalman estimators, we can obtain the information fusion steady-state fused white noise deconvolution estimators, which can reduce the on-line computation burden.

**Theorem 4.** For multisensor time-invariant system (1)-(3), with assumptions 1 and 2, the local steady-state Kalman predictor is given as

\[ \hat{x}_i(t + 1 | t) = \Phi \hat{x}_i(t | t - 1) + B u(t) + K_p e_i(t) \] (52)

\[ e_i(t) = y_i(t) - H_i \hat{x}_i(t | t - 1) \] (53)
or
\[ \hat{x}_j(t + 1 | t) = \Psi_{p_j} \hat{x}_j(t | t - 1) + Bu(t) + K_{p_j} y_j(t) \]
\[ \Psi_{p_j} = \Phi - K_{p_j} H_j \]
\[ K_{p_j} = [(\Phi \Sigma_j H_j^T + \Gamma S_j)] Q_{e_j}^{-1} \]
\[ Q_{e_j} = H_j \Sigma_j H_j^T + R_j, \quad R_j = \overline{H_j} \Gamma \hat{Q} \Gamma^T \hat{H}_j + Q_{e_j} \]
\[ S_j = Q \Gamma^T \hat{H}_j \]
\[ R_j = \overline{H}_j \Gamma \hat{Q} \Gamma^T \hat{H}_j \]

where \( \Sigma_j \) satisfies the steady-state Riccati equation
\[ \Sigma_j = \Phi \Sigma_j \Phi^T - (\Phi \Sigma_j H_j^T + \Gamma S_j) [H_j \Sigma_j H_j^T + R_j]^{-1} \times
\[ \times \Phi \Sigma_j H_j^T + \Gamma S_j] + \Gamma R_j \Gamma^T \]

The steady-state smoothing error variance matrices
\[\hat{\Sigma}_j(t | t + N) = \sum_{k=0}^{N} M_{j}(k) \hat{e}_j(t + k), N \geq 0, i = 1, \ldots, L \]

where we define
\[ M_{j}(0) = S_{j} Q_{e_j}^{-1} \]
\[ M_{j}(k) = D_{j} \Psi_{p_j}^{(k-1)} H_j^T Q_{e_j}^{-1}, k \geq 1 \]
\[ D_{j} = Q \Gamma^T - S_{j} K_{p_j}^{-1} \]

with definition \( \Psi_{p_j}^{(k-1)} = (\Psi_{p_j}^{(k-1)})^{-1} \).

The steady-state smoothing error variance matrices
\[ P_{j}^{w}(N) = Q - \sum_{k=0}^{N} M_{j}(k) Q_{j} M_{j}^T(k), N \geq 0 \]
\[ P_{j}^{w}(N) = Q, \quad N < 0 \]

The steady-state smoothing error cross-covariance matrices are given as
\[ P_{j}^{w}(N) = \Psi \Sigma_j \Psi_{p_j}^T + \sum_{k=0}^{N} [K_{p_j}^w M_{j}(k)] S_{j} Q_{j}^T [K_{p_j}^w]^T \]
\[ \Psi_{p_j}^w = \sum_{k=0}^{N} M_{j}(k) H_j \Psi_{p_j}^{k-1} \]
\[ K_{p_j}^w = \delta_{\rho} I_{\rho} - \sum_{k=\rho+1}^{N} M_{j}(k) H_j \Psi_{p_j}^{k-\rho-1} \Gamma \]
\[ \rho = 0, \ldots, N - 1, \delta_{\rho} = 0, \delta_{0} = 0 (\rho \neq 0) \]
\[ K_{p_j}^w = \sum_{k=0}^{N} M_{j}(k) H_j \Psi_{p_j}^{k-\rho-1} K_{p_j} - M_{j}(\rho) \]
\[ \rho = 0, \ldots, N - 1, K_{p_j}^w \]

Specially, for \( i = j \), we have a formula to compute \( P_{i}(N) \) as follows
\[ P_{i}(N) = \Psi_{iN} \Sigma_{i} \Psi_{iN}^T + \sum_{\rho=0}^{N} [K_{i\rho}^w M_{i}(\rho)] S_{i} Q_{i}^T [K_{i\rho}^w]^T \]

which is different from (65).

**Proof.** For the steady-state Kalman filtering, we have that
\[ K_{p_j}(t) \rightarrow K_{p_j}^w, \quad \Psi_{p_j}(t) \rightarrow \Psi_{p_j}^w, \quad P_{j}(t + 1 | t) \rightarrow \Sigma \]
\[ P_{j}(t + 1 | t) \rightarrow \Sigma_{j}, Q_{ej}(t) \rightarrow Q_{e_j} \], as \( t \rightarrow \infty \). Taking \( t \rightarrow \infty \) in Lemmas 1-2, Theorems 1-3, we directly obtain Theorem 4.

**Theorem 5.** For multisensor time-invariant system (1)-(3) with assumptions 1 and 3, the three distributed fusion steady-state fixed-lag smoother is given by
\[ \hat{w}_{j}(t | t + N) = \sum_{i=1}^{L} \Omega_{ij} \hat{w}_{j}(t | t + N), N \geq 0 \]
\[ \hat{w}_{j}(t | t + N) = \sum_{i=1}^{L} \Omega_{ij} \hat{w}_{j}(t | t + N), N < 0 \]

where \( \hat{w}_{j}(t | t + N) \) are computed via Theorem 4. The weights \( \Omega_{ij} \) are computed via Theorem 3, where \( \Omega_{ij} \rightarrow \Omega, \omega_{ij} \rightarrow \Sigma(t | t + N), P_{j}^{w}(t | t + N) \) and are replaced by \( \Omega, \omega_{ij}, \Sigma, P_{j}^{w}(N) \) and \( P_{j}^{w}(N) \) respectively.

We have the steady-state accuracy relation
\[ tr P_{j}^{w}(N) \leq tr P_{j}^{w}(N) \leq tr P_{0}^{w}(N) \leq tr P_{0}^{w}(N) \leq tr P_{0}^{w}(N) \]
\[ N \geq 0, j = 1, \ldots, L \]

where \( P_{0}^{w}(N) \), \( P_{0}^{w}(N) \) and \( P_{0}^{w}(N) \) denote the steady-state error variance matrix of fusers with matrix weights, diagonal matrix, and scalar weights, respectively.

**Proof.** Taking \( t \rightarrow \infty \) in Theorem 3, we straightforward obtain Theorem 5.

V. SIMULATION EXAMPLES

**Example 1.** Consider the 3-sensor discrete time-varying tracking system with colored measurement noise
\[ x(t + 1) = \Phi(t)x(t) + \Gamma(t)w(t) \]
\[ z_i(t) = \overline{H}_i(t)x(t) + \eta_i(t) \]
\[ \eta_i(t) = A_i(t) \eta_i(t) + \xi_i(t), i = 1, 2, 3 \]
\[ \Phi(t) = \begin{bmatrix} 0.1 & 0.25 \\ 0.1 & 1 \end{bmatrix}, \Gamma(t) = \begin{bmatrix} 2 \cos \frac{2\pi}{200} \\ 0 \end{bmatrix} \]
\[ \overline{H}_i(t) = \begin{bmatrix} 0.3 + 0.1 \cos \frac{2\pi}{100} \end{bmatrix}, A_i(t) = 0.001 + 0.0005 \cos \frac{2\pi}{100} \]
\[ A_2(t) = 0.002 + 0.0005 \cos \frac{2\pi}{100}, A_3(t) = 0.003 + 0.0005 \cos \frac{2\pi}{100} \]

where \( w(t) = (1 + 0.1 \cos \frac{2\pi}{200})b(t)g(t) \) is Bernoulli-Gaussian input white noise[1,2], \( b(t) \) Bernoulli white noise, satisfying \( P(b(t) = 1) = \lambda, P(b(t) = 0) = 1 - \lambda \), \( \lambda = 0.4 \). \( g(t) \) is a Gaussian white noise with zero mean and variance matrix \( \sigma_g^2 = 0.03 \), and is independent of \( b(t) \). \( w(t) \) and \( \xi_i(t) \) are
independent Gaussian white noise with variances white noises with zero means and variance matrices
\( \sigma^2_{\text{white noise}} = 0.001 \), \( \sigma^2_{\text{white noise}} = 0.002 \), \( \sigma^2_{\text{white noise}} = 0.003 \), respectively. The problem is to compare the accuracy of local white noise deconvolution smoothers \( \hat{w}_i(t | t+3) \), fused white noise deconvolution smoother \( \hat{w}_0(t | t+3) \), and centralized fuser
\( \hat{w}_c(t | t+3) \). The simulation results are shown in Fig.1-Fig.8 and Table 1. In Fig.1-Fig.5, we can see the comparison between \( w(t) \) and \( \hat{w}_0(t | t+3) \), \( \theta = 0,1,2,3, \). In Fig.6 and Table 1, we can see that the accuracy of fusion smoothers is higher than that of each local smoother, and the theoretical accuracy relation (50) holds. 300 Monte Carlo runs are carried out and the means square error (MSE) curves is shown in the Fig.7, where the MSE value at the time \( t \) is defined as

\[
\text{MSE}_i(t) = \frac{1}{m} \sum_{j=1}^{m} \hat{w}_i^{(j)}(t | t+3)^T \hat{w}_i^{(j)}(t | t+3) 
\]

(79)

where \( \hat{w}_i^{(j)}(t | t+3) = w(t) - \hat{w}_i^{(j)}(t | t+3) \), \( i = 0,1,2,3, \), \( t = 1,\cdots,100 \), \( \hat{w}_i^{(j)}(t | t+3) \) is the \( j \)th sample of the stochastic process \( \hat{w}_i(t | t+3) \), \( j = 1,\cdots,m \), \( m = 300 \) is the sampled number. From Fig.7, we see that the accuracy of the fuser is higher than that of each local smoother; the accuracy of the centralized fuser is higher than that of the weighted fuser. The comparison between the MSE and the theoretical traces of the estimating error covariance is shown in Fig. 8, from which we can see that the MSE values can be considered as the sampled mean of the theoretical values.

![Fig.1](image1.png)

Fig.1 input white \( w(t) \) and local optimal white noise smoother \( \hat{w}_i(t | t+3) \)

![Fig.2](image2.png)

Fig.2 input white \( w(t) \) and local optimal white noise smoother \( \hat{w}_i(t | t+3) \)

![Fig.3](image3.png)

Fig.3 input white \( w(t) \) and local optimal white noise smoother \( \hat{w}_0(t | t+3) \)

![Fig.4](image4.png)

Fig.4 input white \( w(t) \) and fused smoother weighted by calars \( \hat{w}_0(t | t+3) \)

![Fig.5](image5.png)

Fig.5 input white \( w(t) \) and centralized fused smoother \( \hat{w}_c(t | t+3) \)

![Fig.6](image6.png)

Fig.6 Comparison of local and fused theoretical traces of smoothing error variance matrices

- sensor1
- sensor2
- sensor3
- fusion weighted by scalars
- centralized fusion
Table 1. Comparison of local theoretical traces $\text{tr}P_i(t|t+3)$ and fused theoretical traces $\text{tr}P_0(t|t+3)$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$t=$50</th>
<th>$t=$100</th>
<th>$t=$150</th>
<th>$t=$200</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{tr}P_i(t</td>
<td>t+3)$</td>
<td>0.01125805320188</td>
<td>0.00381120727920</td>
<td>0.01119676525759</td>
</tr>
<tr>
<td>$\text{tr}P_0(t</td>
<td>t+3)$</td>
<td>0.01114353592256</td>
<td>0.00322240817945</td>
<td>0.01107494167349</td>
</tr>
</tbody>
</table>

I. CONCLUSIONS

Based on the optimal information fusion rules weighted by matrices, diagonal matrices and scalars in the linear minimum variance sense, three distributed fused white noise deconvolution smoothers have been presented for the linear discrete time-varying stochastic control systems with multisensor and colored measurement noises. In order to compute the optimal weights, the new formulas computing the cross-covariance among local smoothing errors have been presented. They are locally optimal, and are globally suboptimal. A Monte Carlo simulation examples shown the accuracy relations among local fused white noise deconvolution smoothers. The Monte-Carlo simulation results show that the accuracy distinction of three white noise fusers is not obvious, so that employing the white noise fuser with scalars weights is suitable for real time application. The proposed results overcome the limitations and drawbacks in some references.

ACKNOWLEDGMENT

This work was supported by National Natural Science Foundation of China under Grant NSFC-60374026. The authors wish to thank the reviewers for their constructive comments.

REFERENCES