Steady-state Optimal Weighted Fusion Fractional Kalman Filter
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Abstract: Based on the CARMA innovation model, a steady-state optimal fractional Kalman filter is presented for the linear fractional order discrete state-space systems. It is more universal and practical, which is because the description of fractional order is more accurate than that given by integer order. The steady-state optimal weighted measurement fusion fractional Kalman filter is presented by applying the optimal measurement fusion algorithm to the multisensor linear discrete fractional state-space systems. Compared with centralized fusion Kalman filter, it can reduce the computational burden because of the lower dimension of the measurement vector. It is numerically identical to the centralized fusion Kalman filter, so that it has the global optimality. A simulation example shows its effectiveness.

Index Terms: Discrete fractional state-space systems, fractional Kalman filter, information fusion, weighted measurement fusion

I. INTRODUCTION

In order to meet the accurate needs for system analysis and integration, the fractional calculus started to be more and more interesting [1,2]. The research of fractional calculus on the viscoelasticity is given by [3], and that on the fractals is given by [4]. Using fractional calculus, the control algorithms both in frequency and time domains are given by [5] and [6], respectively. However, the Kalman filtering for the fractional order system is seldom researched [7-9].

The multisensor information fusion has received great attention in recent years due to extensive application backgrounds. The information fusion Kalman filtering has widely been applied in many fields including guidance, defence, robotics, integrated navigation, target tracking, GPS positioning, communication, signal processing, and control [10]. For Kalman filtering-based fusion, two basic fusion methods are centralized and distributed (or weighted) fusion methods, depending on whether raw data are used directly for fusion or not [11]. The centralized fusion method can give the globally optimal state estimation by directly combining local measurement data. But its disadvantages are that it may require a larger computational burden and high data rates for communication. The weighted fusion methods also include the state and measurement weighted fusion methods. The weighted measurement fusion methods directly weight the local sensor measurements to obtain a weighted measurement fusion equation, which accompanies the state equation to yield a weighted measurement fusion Kalman filter [12, 13]. They have global optimality, i.e. they are functionally equivalent to the centralized measurement fusion Kalman filter in certain conditions [12, 13]. They only require a smaller computational burden because of the lower dimension of the fused measurement vector. In [12, 13], the functional equivalence between the weighted measurement fusion method and the centralized fusion method was proved using the information filter method and the method of computing the inverse of partitioned matrix. However, the fusion algorithm has never been applied to the multisensor fractional order systems.

Now, some weighted measurement fusion Kalman filtering algorithms for the integral order system with uncorrelated measurement noises was given in [12, 13]. The correlated measurement fusion Kalman estimators with the same local dynamic model were presented for the multisensor integral order systems in [14]. An optimal distributed fusion steady-state Kalman filter was presented for the multisensor integral order systems with colored measurement noises and different local dynamic model in [15]. However, the fusion filtering problem for the multisensor fractional order systems has not been solved.

In order to overcome the above drawback and limitation, the steady-state optimal fractional Kalman filter is presented for the linear discrete fractional state-space systems. The detail derivation is given. The steady-state optimal weighted measurement fusion fractional Kalman filter is presented by applying the optimal measurement fusion algorithm to the multisensor linear discrete fractional state-space systems. Compared with centralized fusion Kalman filter, it can reduce the computational burden because of the lower dimension of the measurement vector. It is numerically identical to the centralized fusion Kalman filter, so that it has the global optimality. A simulation example shows the performance of the proposed fuser.

II. PROBLEM FORMULATION

Consider the linear fractional order stochastic discrete
state-space system with \( L \) sensors
\[
\Delta^x x(t+1) = \Phi_j x(t) + Bu(t) + w(t)
\] (1)
\[
x(t+1) = \Delta^x x(t+1) - \sum_{j=1}^{\infty} (-1)^j Y_j x(t+1-j)
\] (2)
\[
y_j(t) = H x(t) + v_j(t), \quad i = 1, \ldots, L
\] (3)
where \( t \) is the discrete time, \( \Delta \) is the fractional order, \( x(t) \in \mathbb{R}^n \) is the state, \( y_j(t) \in \mathbb{R}^{m_j} \) is the measurement, \( u(t) \in \mathbb{R}^p \) is the known control input, \( w(t) \in \mathbb{R}^r \) and \( v_j(t) \in \mathbb{R}^{m_j} \) are white noises, and \( \Phi_j, B \) and \( H \) are the constant matrices with compatible dimensions.

Assumption 1 \( w(t) \) and \( v_j(t) \) are independent white noises with zero mean and variances \( Q_w \) and \( Q_{v_j} \):
\[
E\left[\begin{bmatrix} w(t) \\ v_j(t) \end{bmatrix} \begin{bmatrix} w^T(k) \\ v_j^T(k) \end{bmatrix}\right] = \begin{bmatrix} Q_w & 0 \\ 0 & 0 \\ 0, \delta_j \end{bmatrix}
\] (4)

Assumption 2 \( x(0) \) is uncorrelated with \( w(t) \) and \( v_j(t) \), and
\[
E[x(0) - \mu_x] [x(0) - \mu_x]^T = P_0
\] (5)

Assumption 3 The system is completely observable and controllable, or the system is stable [9].

The steady-state optimal measurement fusion Kalman filtering problem is to find the local and weighted fusion linear minimum variance state filters \( \hat{x}_i(t|t) \), \( i = 1, \ldots, L \) and \( \hat{x}(t|t) \) based on the measurements \( y_i(t), y_{1}, \ldots, y_L(t) \).

### III. LOCAL STEADY-STATE OPTIMAL FRACTIONAL KALMAN FILTER

From (1), (2) and (3), we have
\[
y_j(t) = H[I_w - q^{-1} \Phi_j + \sum_{j=1}^{\infty} q^{-j} (-1)^j Y_j] Bu(t-1) + w(t-1) + v_j(t)
\] (6)
where \( I_w \) is a unit matrix, \( q^{-1} \) is the backward shift operator, \( q^{-1} x(t) = x(t-1) \).

Introducing the left-coprime factorization
\[
[I_w - q^{-1} \Phi_j + \sum_{j=1}^{\infty} q^{-j} (-1)^j Y_j, I] = F(q^{-1})A(q^{-1})
\] (7)
\[
A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n}, \quad a_n = 1, \quad a_{n_m} \neq 0
\] (8)
\[
F(q^{-1}) = I_w + F_0 q^{-1} + \cdots + F_k q^{-k}, \quad n_f \leq \mu - 1, \quad F_{n_f} = 0
\] (9)
\[
F_i = \Phi_j F_{i-1} - (-1)^j Y_j F_{i-1 + a_i I}, \quad F_0 = I_w, \quad a_i = 0(i > n_f)
\] (10)
where \( q^{-1}A(q^{-1}) \) is the minimized polynomial for \( \Phi_j \).

Substituting (12) into (11) yields
\[
A(q^{-1})I_w y(t) = HF(q^{-1})[Bu(t-1) + w(t-1)] + A(q^{-1})I_w v_i(t)
\] (11)

Under the assumptions that \( (A(q^{-1})I_w, HF(q^{-1})Bq^{-1}) \), \( HF(q^{-1})y_q \) is left-coprime and \( A(q^{-1})I_w \) and \( HF(q^{-1})y_q \) do not have the left factor with zeros of the determinant on unit circle, applying (15) yields the CARMA innovation model
\[
A(q^{-1})y_i(t) = B(q^{-1})Bu(t) + D(q^{-1})e(t)
\] (12)
where \( B(q^{-1}) = HF(q^{-1})y_q \) is stable, \( D_0 = I_w \), and the innovation \( e(t) \) is a white noise with zero mean and the variance matrix \( Q_\epsilon \), and we have the relationship as follows
\[
D_1(q^{-1})e(t) = B(q^{-1})w(t) + A(q^{-1})I_w v_i(t)
\] (13)
where \( D_1(q^{-1}) \) and \( Q_\epsilon \) can be obtained by the Gevers and Wouters algorithm [17].

**Theorem 1** For the system (1)-(3) with the assumptions 1-3, the steady-state optimal fractional Kalman filter and predictor are given by
\[
\hat{x}_i(t+1|t+1) = \hat{x}_i(t+1|t) + K_i e_i(t+1)
\] (14)
\[
\hat{x}_i(t+1|t) = \Phi_j \hat{x}_i(t+1|t) + Bu(t)
\] (15)
\[
\hat{x}_i(t+1|t) = \hat{x}_i(t+1|t) - \sum_{j=1}^{\infty} (-1)^j Y_j \hat{x}_i(t+1-j) + \hat{x}_i(t+1-j)
\] (16)
and the prediction gain is given by
\[
K_i = \begin{bmatrix} H(F_i - a_i) \\ D_{i-1} - a_i I_w \\ \vdots \\ D_{i-1} - a_i I_w \\ D_{i-1} - a_i I_w \end{bmatrix}
\] (17)

where the pseudo-inverse \( X^+ \) of matrix \( X \) is defined
\[
X^+ = (X^TX)^+X^T
\] (18)

**Proof** Applying Assumption 2 yields that the steady-state Kaman filter and predictor exist. [17].

From (1), (2) and the projective property, we have
\[
\Delta^x \hat{x}_i(t+1|t) = \Phi_j \hat{x}_i(t+1|t) + Bu(t)
\] (19)
\[
\hat{x}_i(t+1|t) = \Delta^x \hat{x}_i(t+1|t) - \sum_{j=1}^{\infty} (-1)^j Y_j \hat{x}_i(t+1-j) + \hat{x}_i(t+1-j)
\] (20)
then we have (19). Applying the projective formula yields (18).
and (20).

Using the Fadeeva formula (12), and applying (18), (19) and (20), we have
\[
A(q^{-1})J_n y(t) = H[F(q^{-1}) - A(q^{-1})]K_p e_i(t) + HF(q^{-1})B(t)u(t) + A(q^{-1})J_n e_i(t)
\]
Comparing (16) and (25) yields
\[
D_k = H(F_i - a_k)K_p + a_k I_n, \quad k = 0, 1, \cdots \beta
\]
where \(D_k = 0, \quad k > n_j\).
From Assumption 3, we have
\[
\text{rank}(H^T, (HF_i)^T, \cdots (HF_{\beta-1})^T)^T = n
\]
Applying the Reduction to Absurdity yields \(n_j \geq \beta - 1\). Then From (26), we have (21). The proof is completed.

IV. MEASUREMENT FUSION STEADY-STATE OPTIMAL FRACTIONAL KALMAN FILTER

Introducing an augmented measurement vector \(y(t)\), we combine all measurement equations to obtain a centralized measurement fusion equation as
\[
y(t) = H^0 x(t) + v(t)
\]
with the definitions
\[
y(t) = [y_1^T(t), \cdots, y_n^T(t)]^T
\]
\[
H^0 = [H^T, \cdots, H^T]^T
\]
\[
v(t) = [v_1^T(t), \cdots, v_n^T(t)]^T
\]
and the variance matrix \(R^0(t)\) is given as follows
\[
R^0(t) = \text{block diag}[R(t), \cdots, R(t)]
\]

For the centralized fusion system (1), (2) and (28), applying the standard Kalman filter [9] with the initial time \(t_0 = 0\), we can obtain the centralized fusion steady-state Kalman predictor \(\hat{x}(t + 1 | t)\) and Kalman filter \(\hat{v}(t)\). They are globally optimal in the sense that their accuracy is higher than that of each local Kalman predictor and filter, and is higher than that of the distributed (weighted) Kalman predictors and filters.

Note that (3) can be considered as L unbiased estimations of \(Hx(t)\) with the local estimation errors \(v_i(t)\) and the local estimation error variance matrices \(R_i(t)\), and we have the measurement model
\[
y(t) = e^T Hx(t) + v(t)
\]
with the definition \(e^T = [I_n, I_n, \cdots I_n]\). So we have the WLS estimator (Gauss-Markov estimator) [16,17] as
\[
y(t) = (e^T R^{-1} e)^{-1} e^T R^{-1} y(t)
\]
where we define \(R^{-1} = [R_0(t)]^{-1}\), applying the definitions of \(e^T, \quad R^0(t)\) and \(y(t)\), we have
\[
y(t) = \sum_{j=1}^{N} R_j^{-1} y_j(t)
\]
which composed of local measurements by weighting. So we have the new fusion measurement equation as
\[
y(t) = Hx(t) + v(t)
\]
where \(v(t)\) is the measurement error for \(Hx(t)\), and it has the minimized measurement error variance matrix [16,17] as
\[
R(t) = \sum_{j=1}^{N} R_j^{-1} (t)]^{-1}
\]
In fact, substituting (33) into (34) yields (36), where
\[
v(t) = \sum_{j=1}^{N} R_j^{-1} (t) v_j(t)
\]
and we have the variance matrix \(R(t)\) of \(v(t)\).
Note that
\[
\sum_{j=1}^{N} R_j^{-1} (t) > R_i(t), \quad i = 1, \cdots, L
\]
and from (37), we have
\[
R(t) = \sum_{j=1}^{N} R_j^{-1} (t)]^{-1} < R_i(t), \quad i = 1, \cdots, L
\]
which shows that the error variance matrix in the weighted measurement fusion equation (36) is less than that in each local measurement equation, i.e. the weighted measurement fusion equation (36) improve the accuracy of each local measurement equation.

Remark 1 For the weighted measurement fusion system (1), (2) and (36), applying the standard Kalman filter [9] with the initial time \(t_0 = 0\), we can obtain the weighted measurement fusion Kalman predictor \(\hat{x}(t + 1 | t)\) and Kalman filter \(\hat{v}(t)\).

Compared with the centralized measurement equation, the measurement equation and measurement vector in (36) are \(m \times n\) and \(m \times 1\) matrices and these in (28) are \(mL \times n\) and \(mL \times 1\) matrices. So the computational burden for weighted measurement fusion Kalman filters is reduced obviously. Moreover, they have the same accuracy as given by theorem 2.

Lemma 1 [18] For the weighted measurement fusion linear discrete fractional order stochastic state-space system (1), (2) and (36) with the assumptions 1-3, the weighted measurement fusion Kalman predictor \(\hat{x}(t + 1 | t)\) and filter \(\hat{v}(t)\) and their error variance matrices are numerically identical to these for the centralized fusion systems (1), (2) and (28), i.e.
\[
\hat{x}(0) (t | t) = \hat{x}(t | t), \quad P^{(0)} (t | t) = P(t | t), \quad \forall t
\]
\[
\hat{x}(0) (t | t-1) = \hat{x}(t | t-1), \quad P^{(0)} (t | t-1) = P(t | t-1), \quad \forall t
\]
with the same initial values
\[
\hat{x}(0) (0 | 0) = \hat{x}(0 | 0), \quad P^{(0)} (0 | 0) = P(0 | 0)
\]
or
\[
\hat{x}(0) (0 | -1) = \hat{x}(0 | -1), \quad P^{(0)} (0 | -1) = P(0 | -1)
\]

V. SIMULATION EXAMPLE

Consider a 3-sensor linear discrete fractional order stochastic state-space system with unit step input
\[
\Delta^\gamma x(t + 1) = \begin{bmatrix} 0 & 0.8 \\ -1 & 0.4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + w(t)
\]
\[
x(t + 1) = \Delta^\gamma x(t + 1) - \sum_{j=1}^{N} (-1)^j \gamma_j x(t + 1 - j)
\]
\[ y_i(t) = [0.1 \ 0.3]x(t) + v_i(t), \quad i = 1, 2, 3 \]  
\[ (47) \]
where \( Y \) is the fractional order, \( x(t) \in R^2 \), \( u(t) \in R \), \( y_i(t) \in R \) are the state, input and output at time \( t \), \( w(t) \in R^2 \) and \( v_i(t) \in R \) are the white noise with zero mean and variance matrices \( Q = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} \) and \( R_i = 0.03, \quad R_i = 0.05 \), \( R_i = 0.07 \) and \( n_1 = 0.9, \quad n_2 = 1.1 \). The problem is to find the local and weighted measurement fusion fractional order Kaman filters \( \hat{x}_i(t | t), \quad i = 1, 2, 3 \) and \( \hat{x}(t | t) \).

\[ \begin{aligned} &\begin{array}{c} \text{(a) The first component of} \\ \text{the original and estimated} \\ \text{state variables} \\ \end{array} \\ &\begin{array}{c} \text{(b) The second component of} \\ \text{the original and estimated} \\ \text{state variables} \\ \end{array} \\ &\text{Fig. 1. The state } x(t) \text{ and the local Kaman filter } \hat{x}_i(t | t) \text{ for sensor 1} \\ &\text{Fig. 2. The state } x(t) \text{ and the local Kaman filter } \hat{x}_i(t | t) \text{ for sensor 2} \\ &\text{Fig. 3. The state } x(t) \text{ and the local Kaman filter } \hat{x}_i(t | t) \text{ for sensor 3} \\ &\text{Fig. 4. The state } x(t) \text{ and the weighted measurement fusion Kaman filter } \hat{x}(t | t) \\ &\text{Fig. 5. The local and weighted measurement fusion filtering error square sum curves} \\ &\text{VI. CONCLUSIONS} \\ &\text{The simulation results are shown in Fig.1- Fig. 5. The curves for the state and the local and weighted fusion Kaman filters are given by Fig. 1- Fig. 4, where the solid lines denote the true value, and the dotted curves denote the estimates. Fig. 5 gives the curves for the local and weighted measurement fusion filtering error square sum. They show that the state variables were estimated with high accuracy, and the weighted measurement fusion algorithm greatly improves the filtering accuracy.} \\ &\text{The fractional Kaman filter is presented for the linear discrete fractional state-space systems. The detail derivation is given. The steady-state optimal weighted measurement fusion fractional Kaman filter is presented by applying the optimal measurement fusion algorithm to the multisensor linear discrete fractional state-space systems. Compared with centralized fusion Kaman filter, it can reduce the computational burden because of the lower dimension of the measurement vector. It is numerically identical to the centralized fusion Kaman filter, so that it has the global optimality. A simulation example shows the performances of the proposed algorithm. The proposed results overcome the limitations and drawbacks in some references.} \]
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